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We demonstrate that secure quantum key distribution systems based on continuous variables implementations can operate beyond the apparent 3 dB loss limit that is implied by the beam splitting attack [6]. The loss limit was established for standard minimum uncertainty states such as coherent states. We show that by an appropriate postselection mechanism we can enter a region where Eve's knowledge falls behind the information shared between Alice and Bob even in the presence of substantial losses.

The distribution of random keys for cryptographic purposes can be made secure by using the fundamental properties of quantum systems such that any interception of the key information can be detected. This was first discussed for discrete systems [1] and experimental demonstrations have been carried out using optical sources, which produce low photon number states [2]. More recently schemes based on continuous quantum variables have been proposed [3–6]. Apart from being of fundamental interest these schemes offer certain practical advantages. However, they all share one major disadvantage: currently it is thought that the use of continuous variable techniques does not allow quantum key distribution (QKD) beyond 50 % loss [6] at least as long as advanced technologies such as entanglement distillation are not available. This severely limits the applicability of such schemes.

The argument leading to this limit is based on an optimal cloning approach for optical signals that corresponds to a beam splitting attack on the signals [6]. At the loss limit an eavesdropper Eve can replace the lossy channel by a perfect one with an adapted beam splitter to mimic the losses. She can then generate a cloned signal with a fidelity which depends on the beam splitter transmission. In order to extract a secure key out of the material with the usual privacy amplification tools, however, the mutual information between Alice and Bob I_{AB} has to exceed the information that each of them shares with Eve: $I_{AB} > \max\{I_{AE}, I_{EB}\}$ [7], which is not fulfilled in the case of 50% or more loss.

On the other hand in certain situations it is still pos-

sible to create a secure key even if this condition is violated. For classical correlations the procedure is known as advantage distillation [7]; upon closer investigation this turns out to be a form of postselection. Postselection is a standard intrinsic procedure in conventional QKD with weak pulses: if no photon is detected by Bob, or when Alice and Bob did not measure in the same basis, the corresponding time slot is ignored and hence does not contribute to the raw data. Without this postselection the condition $I_{AB} > \max\{I_{AE}, I_{EB}\}$ could never be reached for any QKD protocol for losses beyond 3 dB, because then Eve has access to better signals than Bob. However, postselection allows unconditionally secure key exchange in presence of large losses, limited basically only by the photo-detection process [8]. The situation becomes more subtle for continuous variable schemes, since then always a non-vacuum signal reaches Bob and correlations appear between the data measured by Bob and that of an potential eavesdroppers via Alice's state preparation. Thus the postselection has to be made more conscious, and here we show how to do this. The selection of favorable data for Alice and Bob has been previously addressed in the context of implementing the BB84 protocol with weak coherent pulses in the presence of a strong phase reference pulse [9]. Our results demonstrate that continuous variables and weak coherent pulse schemes are closely linked in the basic principles.

We consider the following scheme, which is similar to those proposed by Cerf et al. [5] and Grosshans and Grangier [6]. Alice sends an ensemble of coherent states to Bob with a Gaussian distribution of complex amplitudes centered on the vacuum. Bob measures either of two conjugate quadratures, say for example the in- and out-of phase quadratures X and Y , using homodyne detection. The measurement results x are then given as eigenvalues of quadrature operator $\hat{x}_\lambda = \frac{1}{2}(\hat{a}e^{-i\lambda} + \hat{a}^\dagger e^{i\lambda})$ with $\lambda = 0$, or $\frac{\pi}{2}$. Bob will effectively see a Gaussian distribution of real amplitude coherent states when he looks at the quadrature X and a Gaussian distribution of imaginary amplitude coherent states when he looks at the quadrature Y . Bob reveals which quadrature he measured in each time interval and estimates whether Alice prepared a coherent state with a positive or negative displacement in the corresponding quadrature. Alice and Bob can now interpret positive displacements as logical "0" and negative ones as logical "1".

For our analysis of the security of this scheme we extend this protocol and specify the used states by additional steps. After Bob's publication of his choice of the quadrature, Alice will interpret the state she sent either as a member of the set $\{|-\alpha e^{-i\theta}\rangle, |\alpha e^{i\theta}\rangle\}$, if Bob detected the quadrature X , or $\{|-i\alpha e^{-i\theta}\rangle, |i\alpha e^{i\theta}\rangle\}$ ($\alpha \in \mathbb{R}$), if Bob measured the quadrature Y . She now publishes the values of α and θ . In each case, from Bob's and Eve's perspective, this narrows down the number of possible signals to two, for example $|\alpha e^{i\theta}\rangle$ or $|\alpha e^{-i\theta}\rangle$. Thus Alice and Bob can build up a secret key as before when now the encoding reads more specifically: $|\alpha e^{i\theta}\rangle \rightarrow "0"$, $|\alpha e^{-i\theta}\rangle \rightarrow "1"$ for amplitude measurements and $|i\alpha e^{i\theta}\rangle \rightarrow "0"$, $|-i\alpha e^{-i\theta}\rangle \rightarrow "1"$ for phase measurements. Other choices of signal sets are possible, for example sets with point symmetry, but the choice above turns out to be favorable.

To investigate the secrecy of the key the distribution of Bob's data conditioned on the choice of Alice can be accessed using classical communication. For this purpose Alice and Bob open up complete signal descriptions and measurement results for some randomly chosen transmission events. Thus the statistics of Bob's detected results should mirror Alice's coherent state preparation with expected Gaussian distributions centered according to the complex amplitude displacements.

Eve's first strategy is thus passive intervention via the beamsplitter attack [6]. Eve's intervention is indistinguishable from loss. In the typical loss model Alice's state is transformed as

$$\begin{aligned} |\alpha e^{i\theta}\rangle_B |0\rangle_E \\ \rightarrow |\sqrt{\eta} \alpha e^{i\theta}\rangle_B |\sqrt{1-\eta} \alpha e^{i\theta}\rangle_E \end{aligned} \quad (1)$$

for arbitrary α and θ , where η is the transmission efficiency between Alice and Bob. Alice and Bob are none the wiser but Eve ends up with all the lost signal. It was shown [6], that provided the loss is less than 50% it is still possible for Alice and Bob to distill a secure key when faced with such an attack. We will now show that in fact 50 % loss is not an ultimate limit for secure QKD.

We wish to find a way by which Alice and Bob can postselect a subset of the data for which they have a high mutual information, but for which Eve and Alice do not. Alice and Bob can base their selection procedure on the parameter α and θ characterizing the state preparation and Bob's measurement results x or y . The overall mutual information of Alice and Bob can then be subdivided into different *effective information channels* characterized by the parameters (α, θ, x) (where x stands for value of the measured quadrature X or Y respectively), such that

$$I_{AB}^{tot} = \int_{\alpha, \theta, x} dx d\alpha d\theta p(\alpha, \theta, x) \cdot I_{AB}(\alpha, \theta, x). \quad (2)$$

Similarly the overall information Alice shares with Eve can be composed from all single events as

$$I_{AE}^{tot} = \int_{\alpha, \theta} d\alpha d\theta p(\alpha, \theta) \cdot I_{AE}(\alpha, \theta). \quad (3)$$

Note, that the separable nature of the state of Eq. (1) ensures that there is no correlation between Bob's and Eve's quantum uncertainties. Thus Eve's mutual information with Alice does not depend on Bob's detected outcome x . Furthermore, Eve shares with Bob always less information than with Alice, $I_{BE} < I_{AE}$, and it is sufficient to consider Alice and Eve's information only. Altogether, this allows us to evaluate the knowledge of the different parties separately for all effective information channels and we can restrict our analysis to find suitable parameters (α, θ, x) with $I_{AB}(\alpha, \theta, x) > I_{AE}(\alpha, \theta)$. Since the beam splitting attack and protocol itself are symmetrical in respect to the considered conjugate quadratures, it is also sufficient to investigate only the case of a quadrature measurement X by Bob.

To identify the good effective channels, we calculate the mutual information shared by Alice and Eve. Alice sends a priori pure states. However knowing nothing about Alice's state preparation Eve would have to distinguish between two allowed mixed states characterized by positive or negative displacements in the respective quadrature. So far no general expression for the accessible information is known for non-orthogonal mixed states. For this reason we provide Eve with the additional information about α and θ . As a trade-off Eve has to distinguish for each effective channel between two non-orthogonal *pure* states. Since we have chosen without loss of generality to consider the X quadrature here, these states are $|\alpha e^{i\theta}\rangle$ and $|\alpha e^{-i\theta}\rangle$. In this situation the maximum accessible information is known. It is given as a function of the overlap f of the respective two states [10] in the form

$$\begin{aligned} I_{AE}(\alpha, \theta) &= \\ &= \frac{1}{2}(1 + \sqrt{1 - f^2(\alpha, \theta)}) \log(1 + \sqrt{1 - f^2(\alpha, \theta)}) + \\ &+ \frac{1}{2}(1 - \sqrt{1 - f^2(\alpha, \theta)}) \log(1 - \sqrt{1 - f^2(\alpha, \theta)}). \end{aligned} \quad (4)$$

For an effective channel with parameters α and θ the overlap can be calculated as

$$f(\alpha, \theta) = \langle -\alpha e^{-i\theta} | \alpha e^{i\theta} \rangle = \exp\{-4 \cdot (1 - \eta) \cdot E^2\}, \quad (5)$$

where we defined $E = \alpha \cos(\theta)$. The protocol ensures that the overlap and thus the mutual information of Eve depends only on the *effective amplitude* $E = \alpha \cos(\theta)$. As we will see later, the parameters α and θ enter the mutual information of Alice and Bob only in the same combination. This allows us to consider in the following the parameters (E, x) only.

Next, we calculate the mutual information of Alice and Bob. According to the protocol Bob performs quadrature measurements and decodes the bit value as the sign of the detected displacement. Depending on the signal states $|\pm \alpha e^{\pm i\theta}\rangle$, his outcomes x are distributed corresponding to one of the probability distributions

$$P(x | \pm \alpha e^{\pm i\theta}) = |\langle x_0 | \pm \alpha e^{\pm i\theta} \rangle|^2 = \sqrt{\frac{2}{\pi}} \exp\{-2(x \mp \sqrt{\eta} E)^2\} \quad (6)$$

with $|x_0\rangle$ as the eigenstate of the quadrature operators with $\lambda = 0$. This decoding leads to an error rate

$$p_e = \begin{cases} \frac{P(x | -\alpha e^{-i\theta})}{P(x | \alpha e^{i\theta}) + P(x | -\alpha e^{-i\theta})} & \text{for } x > 0 \\ \frac{P(x | \alpha e^{i\theta})}{P(x | \alpha e^{i\theta}) + P(x | -\alpha e^{-i\theta})} & \text{for } x < 0 \end{cases} \quad (7)$$

To illustrate Bob's decoding we consider in Fig. 1 the case of 50% loss and an effective amplitude of $E = 1$. The left graph depicts the two possible distributions for Bob's results x corresponding to Alice's states $|\alpha e^{i\theta}\rangle$ and $|\alpha e^{-i\theta}\rangle$. For the intersection point at $x = 0$, Bob's measurement delivers no information, but for all other values of x , the distribution corresponding to Alice's encoded bit value is more probable. In the right graphs we plotted the probability that Alice has encoded a "0" or "1", if Bob actually obtained the value x . With the arranged bit assignment the smaller value of these graphs gives Bob's error probability corresponding to Eq. (7). Alice's and Bob's mutual information can now be calculated separately for all effective information channels with (x, E) by the Shannon formula

$$I_{AB}(x, E) = 1 + p_e \log_2 p_e + (1 - p_e) \log_2 (1 - p_e) \quad (8)$$

We are now in a position to identify those effective information channels with $I_{AB}(x, E) > I_{AE}(x, E)$, which allow to extract a secret key. Fig. 2 displays the differences of the respective information plotted for the events (E, x) again in the case of 50% loss. Positive valued areas, indicating region of possible secure key exchange, are colored bright, negative ones in dark. Thus our investigation allows us to model an ideal postselection procedure, where all events (x, E) with $I_{AB} > \max\{I_{AE}, I_{EB}\}$ actually contribute to the key.

The comparison of Fig. 2 between the mutual information of Alice and Bob and the information they share with Eve displays an insight that was first recognized in [9]. Alice and Bob can postselect statistically occurring events with large x to decrease the error rate for the shared bit values. On the other hand, Eve's error rate for the selected data is unchanged, because her conditional state is uncorrelated to Bob's results x . Furthermore there exist for each transmission η an optimum effective displacements E , such that the mutual information between Alice and Bob is maximized.

We can evaluate the key rates R_k , that can be achieved in the presented postselection process as

$$R_k = R_r \times \int_S dx dE p(x, E) \cdot [I_{AB}(x, E) - I_{AE}(x, E)], \quad (9)$$

where R_r is the raw data rate, and S denotes the subset of selected effective channels. For the presented protocol the probability $p(x, E)$ that the effective channel is used is composed of Gaussian distribution of the effective amplitude E and the distribution of x conditioned on E . With a distribution width d for the effective amplitudes we find

$$p(x, E) = \sqrt{\frac{2}{d\pi}} e^{-2 \cdot E^2/d} \frac{1}{2} (P(x | \alpha e^{i\theta}) + P(x | -\alpha e^{-i\theta})) \quad (10)$$

with $P(x | \pm \alpha e^{i\theta})$ given in Eq. (6).

First numerical calculations, where we limited our integration over the data set within ± 4 , indicate that in the presence of 50% loss and for an optimized parameter of $d = 2.1$, bit rates up to $R_k = R_r \times 0.0667$ are achievable. For a loss rate of 75%, we have found a key rate of $R_k = R_r \times 0.0073$. These calculations show that the high repetition rates of continuous variable technology, which is expected to be in the GHz region, can actually lead to secure key rates that are well above currently implemented schemes.

We have shown that continuous variable QKD using coherent states in the presence of losses above 50% can still be implemented securely against an eavesdropper using an individual beam splitter attack. The postselection process solely relies on classical data processing and thus does not require sophisticated quantum resources other than coherent states. An absolute proof of security would require the analysis of a more general attack by Eve. However, it is likely that the beam splitting attack is the optimal attack even in this protocol utilizing postselection. The existence of optimum effective displacements for the mutual information between Alice and Bob opens the possibility to construct more elaborated protocols with modified probability distributions for Alice's state preparation to achieve higher the bit rates. However, in all such protocols one has to be cautious to ensure that the beam splitting attack remains the best eavesdropping strategy.

Our result pushes continuous variable QKD closer to practical applications. In addition we have also shown that the type of postselection protocol considered here can also be employed for schemes using squeezed light, such as in [4].

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FIG. 1. Illustration of the error rate of Bob's decoding scheme, here for a loss of 50% and an effective amplitude $E = \alpha \cos(\theta) = 1$. Left: probability distributions of Bob for prepared bits "0" and "1"; Right: probability that Alice has encoded "0" or "1", if Bob obtained a result x .

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- [1] S. Wiesner, *Sigact News*, **15**, 78 (1983); C. H. Bennett and G. Brassard, *Proceedings of IEEE International Conference on Computers, Systems and Signal Processing*, Bangalore, India, December 1984, pp. 175-179; C. H. Bennett, *Phys. Rev. Lett.* **68**, 3121 (1992); A. K. Ekert, *Phys. Rev. Lett.* **67**, 661 (1991).
 - [2] W. T. Buttler, R. J. Hughes, P. G. Kwiat, G. G. Luther, G. L. Morgan, J. E. Nordholt, C. G. Peterson, and C. M. Simmons, *Phys. Rev. A* **57**, 2379 (1998); H. Zbinden, H. Bechmann-Pasquinucci, N. Gisin and G. Ribordy, *Appl. Phys. B* **67**, 743 (1998); P. D. Townsend *Opt. Lett.* **20**, 1048 (1998); M. Bourennane, F. Gibson, A. Karlsson, A. Hening, P. Jonsson, T. Tsegaye, D. Ljunggren, and E. Sundberg, *Opt. Express*, **4**, 383 (1999).
 - [3] T. C. Ralph, *Phys. Rev. A* **61** 010303(R) (1999); M. Hillery, *Phys. Rev. A* **61** 022309 (2000); T. C. Ralph, *Phys. Rev. A* **62**, 062306 (2000); M. D. Reid, *Phys. Rev. A* **62** 062308 (2000); K. Bencheikh, T. Smyul, A. Jankovic and J. A. Levenson, *J. Mod. Opt.* **48**, 1903 (2001); D. Gottesman and J. Preskill, *Phys. Rev. A* **63**, 022309 (2001); A. C. Funk and M. G. Raymer, *Phys. Rev. A* **65**, 042307 (2002).
 - [4] Ch. Silberhorn, N. Korolkova, and G. Leuchs, *Phys. Rev. Lett.*, **88**, 167902 (2002).
 - [5] N. J. Cerf, M. Levy, G. Van Assche, *Phys. Rev. A* **63**, 052311 (2001).
 - [6] F. Grosshans and P. Grangier, *Phys. Rev. Lett.* **88**, 057902 (2002).
 - [7] U. M. Maurer, *IEEE Trans. Inf. Theo.* **39**, 1733 (1993); Ch. B. Bennett, G. Brassard, C. Crepeau, U. M. Maurer, *IEEE Trans. Inf. Theo.* **41**, 1915 (1995).
 - [8] H. Inamori, N. Lütkenhaus, D. Mayers, *quant-ph/0107017* (2001).
 - [9] T. Hirano, T. Konishi, and R. Namiki, *quant-ph/0008037* (2000); in extended abstract of EQIS 2001, Sept. 6-8, 2001, Tokyo, Japan.
 - [10] L. B. Levitin, in *Quantum Communication and Measurements*, V. P. Belavkin, O. Hirota, and R. L. Hudson, eds., Plenum Press, New York (1995), pp 439-448; C. A. Fuchs, *Proceedings of the Fourth International Conference on Quantum Communication, Measurement, and Computing, 1998*, Kluwer Academic, New York (2000), pp. 11-16.

FIG. 2. Comparison of the mutual information between Alice and Bob and the information they share with Eve for different state preparations with effective amplitude $E = \cos(\alpha)$ and measured outcomes x of Bob. Positive areas, colored bright, indicate effective information channels that contribute to secure key exchange since they satisfy $I_{AB}(x, E) > I_{AE}(x, E)$.